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Tension, bending, and flexure of functionally graded cylinders

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Abstract

The classical St. Venant problems (tension, bending and flexure) for isotropic elastic prismatic bars with the elastic moduli varying across the cross-section are examined. Inequalities relating the appropriate effective overall Young's modulus to averages of the actual moduli are derived. The strain energy density for a composite with N elastic phases is examined, and it is found that the strain energy density and thus the elastic moduli are convex functions of the volume fractions. This result is then used to show that, in simple tension, the effective Young's modulus is a minimum for the homogeneous distribution of the phases. It is also shown that, in bending and flexure, the effective Young's modulus can be increased by concentrating the elastic components with the greater Young's modulus further from the axis of bending.

Keywords: Functionally graded materials; Inhomogeneous elasticity; Composites; Effective elastic moduli; St. Venant problems

1. Introduction

The concept of functionally graded materials (FGMs), i.e. composites with smoothly varying constitutive properties, was first suggested by Niino and coworkers at the National Aerospace Laboratory in Japan (Koizumi, 1992; Niino et al., 1987). The original idea was to manufacture super-heat-resistant components for use in the engines and airframe of a supersonic plane, combining the heat resistance of ceramics with the structural properties of metals, an optimal non-homogeneous distribution of the second phase ceramic material was to be employed in this context. The important difference between design with FGMs and conventional materials is that in FGMs, the designer can change the constitutive moduli in the manufacturing in order to optimize whatever function the part is required to do. In order to obtain a theoretical framework, from which FGM structures could be analyzed and designed, advances in two areas are required: (1) The relationship between the microstructure and the macroscopic behavior and (2) the analysis of non-homogeneous graded structures. The first is an extremely active area of research and there are many competing theories in the literature (Benveniste, 1987; Christensen, 1990; Ferrari, 1994). On the second issue, there had been relatively little investigation until recently. With the advent of FGMs though

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there has been a renewed interest in inhomogeneous elasticity. For example, Erdogan and coworkers have analyzed the various problems associated with cracks in inhomogeneous bodies (Delale and Erdogan, 1988; Erdogan and Ozturk, 1992; Erdogan, 1995). Ferrari has considered some basic elastic and thermoelastic solutions, (Ferrari, 1992; Lutz and Ferrari, 1993; Rooney and Ferrari, 1995). Other authors have approached the problem by considering representative volume elements that are themselves graded in some sense (Aboudi et al., 1995; Zuiker and Dvorak, 1994,; Reiter et al., 1997) and there have been studies of microstructural optimization (Nadeau and Ferrari, 1998).

In the context of homogeneous elasticity, the problems associated with St. Venant's name, the deformation of cylinders by forces applied at their ends, has a long history (Casey and Kaplan, 1997), but the inhomogeneous St. Venant problems were first considered in the early 1960s. The torsion problem was formulated in terms of a single stress function by Ely and Zienkiewicz (1960). As in the homogeneous problem, the solution depends only on the shear modulus. The solution for flexure by a transverse load, for cylinders with a constant Poisson's ratio, was investigated by Schile (1962) and Rooney and Ferrari (1995). The general flexure problem, for a non-constant Poisson's ratio, was considered by Reissner (1964), who pointed out that St. Venant's original assumption, that $\tau_{xx} = \tau_{xy} = \tau_{yy} = 0$, is not compatible with the existence of displacements. The flexure problem in general involves auxiliary bending and torsion components. This feature of flexure was noted by Mushkhelishvili (1963) in his consideration of the flexure of compound bars with differing Poisson's ratios. Schile and Sierakowski (1965) reduced the compatibility equations involving the six Beltrami functions to two independent stress functions. This representation was then applied by Rooney and Ferrari (1999) to the St. Venant problems of circular cylinders.

In this article, lower bounds on the effective moduli in terms of averages of the elastic moduli are derived. Then, it is shown, in the context of Hill's concept of representative volume elements, that the strain energy density and thus the elastic moduli must be convex functions of the volume fractions. These two results are then used to prove, for cylinders with graded cross-sections, that the homogeneous distribution gives a minimum for the effective Young's modulus and that in bending and flexure that concentrating the phases with larger Young's moduli away from the axis of bending leads to an effective Young's modulus greater than in the homogeneous case.

2. Tension, bending and flexure of a cylinder

Consider an isotropic elastic cylinder of a constant cross-section Ω under the action of forces acting on the ends. Introduce a Cartesian coordinate system with the z axis parallel to the generators of the cylinder, and assume the elastic moduli depend on x and y only.

Assume that the stress tensor can be written in the form,

$$\tau = \tau^{(0)}(x,y) + z\tau^{(1)}(x,y) \tag{1}$$

with $\tau^{(1)}_{yz} = \tau^{(1)}_{xz} = 0$.

The constitutive equations give the strain in the form,

$$\mathbf{e} = \mathbf{e}^{(0)}(x,y) + z\mathbf{e}^{(1)}(x,y) \tag{2}$$

with $e^{(1)}_{yz} = e^{(1)}_{xz} = 0$.

Choose the following stress function representation,

$$\tau_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial y \partial x}, \quad \tau_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \tag{3}$$

$$\tau_{yz} = \frac{\partial \Xi}{\partial x}, \quad \tau_{xz} = \frac{\partial \Xi}{\partial y} - \int \tau^{(1)}_{zz}\, \mathrm{d}x + f(y) \tag{4}$$

with $\Phi = \Phi^{(0)}(x,y) + z\Phi^{(1)}(x,y)$, $\Xi = \Xi(x,y)$, and f an arbitrary function of y. The six compatibility equations reduce to

$$\frac{\partial^2 e_{zz}}{\partial x^2} = \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{zz}}{\partial x\,\partial y} = 0, \tag{5}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial e_{xx}}{\partial y} - \frac{\partial e_{xy}}{\partial x}\right), \tag{6}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y}\right) = -\frac{\partial}{\partial z}\left(\frac{\partial e_{yy}}{\partial x} - \frac{\partial e_{xy}}{\partial y}\right), \tag{7}$$

$$2\frac{\partial^2 e_{xy}}{\partial x\,\partial y} = \frac{\partial^2 e_{yy}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial y^2}. \tag{8}$$

Eq. (5) implies

$$e_{zz} = A_0 + B_0 x + C_0 y + z(A_1 + B_1 x + C_1 y) \tag{9}$$

with $A_0, B_0, C_0, A_1, B_1, C_1$ arbitrary constants. Substituting this into the constitutive equations leads to

$$\tau_{zz} = 2\mu(1+v)e_{zz} + v\nabla^2\Phi, \tag{10}$$

where μ, v are the shear modulus and Poisson's ratio, respectively. Eqs. (6) and (7) become

$$\mathbf{\nabla}\cdot\left[\frac{1}{\mu}\mathbf{\nabla}\left(\mathbf{\nabla}\Phi^{(1)}\right)\right] - \mathbf{\nabla}\left(2ve_{zz} + \frac{v}{\mu}\nabla^2\Phi^{(1)}\right) = \mathbf{\nabla}\times(\zeta\mathbf{k}), \tag{11}$$

where

$$\zeta = \mathbf{\nabla}\cdot\left(\frac{1}{\mu}\mathbf{\nabla}\Xi\right) - \frac{\partial}{\partial y}\left(\frac{1}{\mu}\int \tau_{zz}^{(1)}\,\mathrm{d}x\right) + \frac{\partial}{\partial y}\left(\frac{f(y)}{\mu}\right). \tag{12}$$

The other compatibility Eq. (8) leads to

$$\mathbf{\nabla}\cdot\left(\mathbf{\nabla}\cdot\left[\frac{1}{\mu}\mathbf{\nabla}(\mathbf{\nabla}\Phi)\right]\right) - \nabla^2\left(\frac{v}{\mu}\nabla^2\Phi\right) = \nabla^2(2ve_{zz}). \tag{13}$$

The condition that the lateral surface be free of traction reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Phi}{\partial x}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Phi}{\partial y}\right) = 0 \quad \text{on } \partial\Omega, \tag{14}$$

$$\frac{\mathrm{d}\Xi}{\mathrm{d}s} = \left[\int \tau_{zz}^{(1)}\,\mathrm{d}x - f(y)\right]\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{15}$$

So, if Ω is simply connected, without loss of generality, let

$$\Phi = \mathbf{\nabla}\Phi = 0 \quad \text{on } \partial\Omega. \tag{16}$$

2.1. Simple tension

For simple tension, the longitudinal strain e_{zz} is constant and the stress functions must be of the form,

$$e_{zz} = A_0, \quad \Phi = A_0\Phi_1(x,y), \qquad \Xi = 0. \tag{17}$$

On the ends, the stress fields must satisfy

$$\int_{\Omega} \mathbf{t}\,\mathrm{d}A = T\mathbf{k}, \quad \int_{\Omega} \mathbf{r} \times \mathbf{t}\,\mathrm{d}A = \mathbf{0}, \tag{18}$$

where $\mathbf{t}$ is the traction vector at the end of the cylinder. Then, the effective Young's modulus for the inhomogeneous cylinder is given by

$$E_{\mathrm{T}} = \frac{T}{e_{zz}A} = \frac{1}{A}\int_{\Omega} \left(2\mu(1+v) + v\nabla^2\Phi_1\right)\mathrm{d}A. \tag{19}$$

The only non-trivial compatibility equation in this case is Eq. (13), and so, Φ_1 satisfies

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \left[\frac{1}{\mu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\Phi_1)\right]\right) - \nabla^2\left(\frac{v}{\mu}\nabla^2\Phi_1\right) = \nabla^2(2v), \tag{20}$$

and A is the area of Ω. The effective Young's modulus in simple tension will be given by Eq. (19)

$$E_{\mathrm{T}} = \frac{1}{A}\int_{\Omega} \left(E + v\nabla^2\Phi_1\right)\mathrm{d}A, \tag{21}$$

where E and v are Young's modulus and Poisson's ratio. Multiplying Eq. (20) by Φ_1 and integrating over Ω gives

$$\int_{\Omega} \Phi_1\nabla^2(2v)\,\mathrm{d}A = \int_{\Omega}\left\{\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \left[\frac{1}{\mu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\Phi_1)\right]\right) - \nabla^2\left(\frac{v}{\mu}\nabla^2\Phi_1\right)\right\}\Phi_1\,\mathrm{d}A. \tag{22}$$

The use of Green's identities and the boundary conditions leads to

$$\int_{\Omega} 2v\nabla^2\Phi_1\,\mathrm{d}A = \int_{\Omega}\left\{\frac{1}{\mu}\mathrm{tr}\left[(\boldsymbol{\nabla}(\boldsymbol{\nabla}\Phi_1))^2\right] - \frac{v}{\mu}\left(\nabla^2\Phi_1\right)^2\right\}\mathrm{d}A. \tag{23}$$

Let

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\Phi_1) = \mathbf{D} + \tfrac{1}{2}\nabla^2\Phi_1\mathbf{1}, \quad \text{where} \quad \mathrm{tr}\,\mathbf{D} = 0 \text{ and } \mathbf{1} \text{ is the identity tensor.} \tag{24}$$

Then,

$$\int_{\Omega} 2v\nabla^2\Phi_1\,\mathrm{d}A = \int_{\Omega}\left\{\frac{1}{\mu}\mathrm{tr}\left[\mathbf{D}^2\right] + \frac{1}{2\mu}(1-2v)\left(\nabla^2\Phi_1\right)^2\right\}\mathrm{d}A. \tag{25}$$

The right-hand side of this equation is positive, so from Eq. (21), we can deduce

$$E_{\mathrm{T}} > \frac{1}{A}\int_{\Omega} E\,\mathrm{d}A. \tag{26}$$

2.2. Pure bending

For bending by a moment of magnitude M in the x direction, we assume the longitudinal strain and the stress functions in the form,

$$e_{zz} = B_0x, \quad \Phi = B_0\Phi_2(x,y), \quad \Xi = 0. \tag{27}$$

The stress field has to satisfy

$$\int_{\Omega} \mathbf{t}\,\mathrm{d}A = \mathbf{0}, \quad \int_{\Omega} \mathbf{r} \times \mathbf{t}\,\mathrm{d}A = M\mathbf{i}. \tag{28}$$

By analogy with the homogeneous bending problem, we can define an effective Young's modulus in bending by

$$E_{\mathrm{B}} = \frac{M}{B_0 I_y} = \frac{1}{I_y} \int_{\Omega} \left(2\mu(1+v)x + v\nabla^2 \Phi_2\right) x \, \mathrm{d}A, \tag{29}$$

where Φ_2 satisfies

$$\nabla \cdot \left(\nabla \cdot \left[\frac{1}{\mu} \nabla(\nabla \Phi_2) \right] \right) - \nabla^2 \left(\frac{v}{\mu} \nabla^2 \Phi_2 \right) = \nabla^2 (2vx), \tag{30}$$

$$I_y = \int_{\Omega} x^2 \, \mathrm{d}A. \tag{31}$$

In a manner analogous to the previous section, it can be proved that

$$\int_{\Omega} v \nabla^2 \Phi_2 x \, \mathrm{d}A > 0, \tag{32}$$

and hence,

$$E_{\mathrm{B}} > \frac{1}{I_y} \int_{\Omega} E x^2 \, \mathrm{d}A. \tag{33}$$

2.3. Flexure by a load along a principal axis

For flexure by a force of magnitude P in the x direction, assume

$$e_{zz} = B_1 xz, \qquad \Phi = z B_1 \Phi_3(x, y), \qquad \Xi \neq 0. \tag{34}$$

The stress field must satisfy the end conditions,

$$\int_{\Omega} \mathbf{t} \, \mathrm{d}A = P\mathbf{i}, \qquad \int_{\Omega} \mathbf{r} \times \mathbf{t} \, \mathrm{d}A = \mathbf{0}. \tag{35}$$

The end conditions imply that

$$P = \int_{\Omega} \tau_{zz}^{(1)} x \, \mathrm{d}A. \tag{36}$$

Thus, we can define the effective Young's modulus for flexure as

$$E_{\mathrm{F}} = \frac{P}{B_1 I_y} = \frac{1}{I_y} \int_{\Omega} \left(2\mu(1+v)x + v\nabla^2 \Phi_3\right) x \, \mathrm{d}A, \tag{37}$$

where Φ_3 satifies Eq. (30) and I_y is given by Eq. (31). Comparing Eqs. (29) and (37) leads to the conclusion that

$$E_{\mathrm{B}} = E_{\mathrm{F}}. \tag{38}$$

3. Strain energy density and convexity

Consider a volume $\mathscr{V}$ of a body consisting of N different elastic materials perfectly bonded together. Let $\mathscr{V}_i$ be the volume of the ith phase, then define the volume fraction of the ith phase, c_i by

$$c_i = \frac{\mathrm{vol}(\mathscr{V}_i)}{\mathrm{vol}(\mathscr{V})}. \tag{39}$$

Initially, assume that the phases are homogeneously distributed. The concept of a representative volume element (RVE) was introduced by Hill (1963), he defined an RVE, to be a portion of the material that is structurally typical of the whole mixture on average and contains a sufficient number of inclusions so that the apparent overall moduli are independent of the surface values of the traction and displacement. Let V be a RVE, and let it be subjected to surface displacements of the kind that would produce a uniform strain in a homogeneous material. This is also the average strain in the inhomogeneous mixture. Let the displacement be

$$\mathbf{u} = \mathbf{E}_0 \mathbf{X} \quad \text{on } \partial V, \tag{40}$$

where $\mathbf{X}$ is the position vector and $\mathbf{E}_0$ is the constant average strain tensor. Let V_i be the volume of the ith phase contained in V. The strain energy density is then

$$U(c_i) = \frac{1}{\mathrm{vol}(V)} \sum_{i=1}^{N} \int_{V_i} \frac{1}{2} \mathbf{E}_i \cdot (\mathbf{C}^{(i)} \mathbf{E}_i)\, \mathrm{d}V, \tag{41}$$

where $\mathbf{C}^{(i)}$ is the elasticity tensor for the ith phase. The various effective moduli can then be calculated by suitable choices of $\mathbf{E}_0$ from

$$U(c_i) = \tfrac{1}{2}\mathbf{E}_0 \cdot (\mathbf{C}(c_i)\mathbf{E}_0). \tag{42}$$

If $\breve{V} \subset V$, then $\breve{V}$ will be an RVE and the strain energy density calculated using $\breve{V}$ should be the same as using V, i.e.,

$$U(c_i) = \frac{1}{\mathrm{vol}(\breve{V})} \sum_{i=1}^{N} \int_{\breve{V}_i} \frac{1}{2} \mathbf{E}_i \cdot (\mathbf{C}^{(i)} \mathbf{E}_i)\, \mathrm{d}V, \tag{43}$$

where

$$c_i = \frac{\mathrm{vol}(\breve{V}_i)}{\mathrm{vol}(\breve{V})} = \frac{\mathrm{vol}(V_i)}{\mathrm{vol}(V)}. \tag{44}$$

Equating the two expressions gives

$$\sum_{i=1}^{N} \int_{\breve{V}_i} \frac{1}{2} \mathbf{E}_i \cdot (\mathbf{C}^{(i)} \mathbf{E}_i)\, \mathrm{d}V = \frac{\mathrm{vol}(\breve{V})}{\mathrm{vol}(V)} \sum_{i=1}^{N} \int_{V_i} \frac{1}{2} \mathbf{E}_i \cdot (\mathbf{C}^{(i)} \mathbf{E}_i)\, \mathrm{d}V. \tag{45}$$

Consider

$$\alpha U(\tilde{c}_i) + (1-\alpha) U(\hat{c}_i) = \frac{\alpha}{\mathrm{vol}(V)} \sum_{i=1}^{N} \int_{V_i} \frac{1}{2} \tilde{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)} \tilde{\mathbf{E}}_i)\, \mathrm{d}V + \frac{1-\alpha}{\mathrm{vol}(V)} \sum_{i=1}^{N} \int_{V_i} \frac{1}{2} \hat{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)} \hat{\mathbf{E}}_i)\, \mathrm{d}V. \tag{46}$$

Using Eq. (45), this becomes

$$\alpha U(\tilde{c}_i) + (1-\alpha) U(\hat{c}_i) = \frac{1}{\mathrm{vol}(V)} \left[\sum_{i=1}^{N} \int_{\tilde{V}_i^{\alpha}} \frac{1}{2} \tilde{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)} \tilde{\mathbf{E}}_i)\, \mathrm{d}V + \sum_{i=1}^{N} \int_{\hat{V}_i^{\alpha}} \frac{1}{2} \hat{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)} \hat{\mathbf{E}}_i)\, \mathrm{d}V \right], \tag{47}$$

where

$$\mathrm{vol}(\tilde{V}_i^{\alpha}) = \alpha\, \mathrm{vol}(V_i); \qquad \mathrm{vol}(\hat{V}_i^{\alpha}) = (1-\alpha)\, \mathrm{vol}(V_i). \tag{48}$$

This can then be expressed in the form,

$$\alpha U(\tilde{c}_i) + (1-\alpha)U(\hat{c}_i) = \frac{1}{\mathrm{vol}(V)}\left[\sum_{i=1}^{N}\int_{\bar{V}_i^{\alpha}} \frac{1}{2}\acute{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)}\acute{\mathbf{E}}_i)\,\mathrm{d}V\right], \tag{49}$$

where

$$\bar{V}_i^{\alpha} = \tilde{V}_i^{\alpha} \cup \hat{V}_i^{\alpha}, \qquad \acute{\mathbf{E}}_i = \tilde{\mathbf{E}}_i \quad \text{on } \tilde{V}_i^{\alpha}, \qquad \acute{\mathbf{E}}_i = \hat{\mathbf{E}}_i \quad \text{on } \hat{V}_i^{\alpha}. \tag{50}$$

Let

$$\bar{c}_i = \alpha\tilde{c}_i + (1-\alpha)\hat{c}_i, \tag{51}$$

then

$$\mathrm{vol}(\bar{V}_i^{\alpha}) = \bar{c}_i\,\mathrm{vol}(V). \tag{52}$$

If $\bar{\mathbf{E}}_i$ is the actual strain in the ith phase corresponding to the given loading with volume fractions $\bar{c}_i$, then the theorem of minimum potential energy, Gurtin (1984) gives

$$\sum_{i=1}^{N}\int_{\bar{V}_i^{\alpha}} \acute{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)}\acute{\mathbf{E}}_i)\,\mathrm{d}V \geqslant \sum_{i=1}^{N}\int_{\bar{V}_i^{\alpha}} \bar{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)}\bar{\mathbf{E}}_i)\,\mathrm{d}V. \tag{53}$$

So,

$$\alpha U(\tilde{c}_i) + (1-\alpha)U(\hat{c}_i) \geqslant \frac{1}{\mathrm{vol}(V)}\left[\sum_{i=1}^{N}\int_{\bar{V}_i^{\alpha}} \frac{1}{2}\bar{\mathbf{E}}_i \cdot (\mathbf{C}^{(i)}\bar{\mathbf{E}}_i)\,\mathrm{d}V\right] = U(\bar{c}_i). \tag{54}$$

This means that the strain energy density is a convex function of the volume fractions.

If the volume fractions are now allowed to vary with position, define an RVE at $\mathbf{r}$ to be a volume $V_{\mathbf{r}}$ that is structurally typical of the material around $\mathbf{r}$ with the following properties:

$$\int_{V_{\mathbf{r}}} \mathbf{r}'\,\mathrm{d}V' = \mathbf{r}, \qquad \int_{V_{\mathbf{r}}} c_i(\mathbf{r}')\,\mathrm{d}V' = c_i(\mathbf{r}). \tag{55}$$

The strain energy density is then

$$U[c_i(\mathbf{r})] = \frac{1}{\mathrm{vol}(V_{\mathbf{r}})}\sum_{i=1}^{N}\int_{V_{\mathbf{r}}^{i}} \mathbf{E}_i \cdot (\mathbf{C}^{(i)}\mathbf{E}_i)\,\mathrm{d}V, \tag{56}$$

where

$$\mathrm{vol}(V_{\mathbf{r}}^{i}) = c_i(\mathbf{r})\,\mathrm{vol}(V_{\mathbf{r}}). \tag{57}$$

An argument similar to the earlier derivation establishes that in this case too, the strain energy density and thus the effective moduli are convex functions of the volume fractions.

4. Multiphase composites

Consider a composite cylinder, with a constant cross-section Ω, made up of N different elastic materials bonded together perfectly, under the action of end loads. Introduce a cartesian coordinate system with the z axis parallel to the generators of the cylinder. If c_i is the volume fraction of the ith phase, assume that $c_i = c_i(x, y)$. If the overall volume fraction of the ith phase is $c_i^{(0)}$, then

$$\int_{\Omega} c_i(x, y)\,\mathrm{d}A = c_i^{(0)}A, \tag{58}$$

where A is the area of Ω. Let $E(c_i)$ be Young's modulus of the composite and let

$$E_0 = E(c_i^{(0)}). \tag{59}$$

Consider the difference between the effective Young's modulus in tension given by Eq. (19) and E_0,

$$E_\mathrm{T} - E_0 \geqslant \frac{1}{A}\int_\Omega (E - E_0)\,\mathrm{d}A \tag{60}$$

using Eq. (26). But from the previous section, E is a convex function of the c_i, which implies (Rockafellar, 1970)

$$E - E_0 \geqslant \sum_{i=1}^{N} \frac{\partial E}{\partial c_i}(c_i^{(0)})\left(c_i(x,y) - c_i^{(0)}\right). \tag{61}$$

Therefore, Eq. (60) becomes

$$E_\mathrm{T} - E_0 \geqslant \frac{1}{A}\sum_{i=1}^{N} \frac{\partial E}{\partial c_i}(c_i^{(0)})\int_\Omega \left(c_i(x,y) - c_i^{(0)}\right)\mathrm{d}A = 0. \tag{62}$$

So the homogeneous distribution of the elastic phases renders the effective Young's modulus in simple tension a minimum.

Now, consider bending of a cylinder made up of two phases. For definiteness, let the volume fraction of the phase with the greater Young's modulus be $c(x,y)$. Then, the volume concentration of the second phase is $1-c$ and E is a convex function of c. Then, in view of Eq. (33), the difference between the effective Young' modulus and E_0 satisfies

$$E_\mathrm{B} - E_0 \geqslant \frac{1}{I_y}\int_\Omega x^2(E - E_0)\,\mathrm{d}A. \tag{63}$$

From a remark of Hill (1963, p. 370), as the first phase has the greater Young's modulus, it follows that E is an increasing function of c. Then,

$$E_\mathrm{B} - E_0 \geqslant \frac{1}{I_y}E'(c_0)\int_\Omega x^2(c(x,y) - c_0)\,\mathrm{d}A, \tag{64}$$

where c_0 is the overall volume fraction of the first phase. If the first phase is concentrated away from the y axis, then

$$\int_\Omega x^2(c(x,y) - c_0)\,\mathrm{d}A \geqslant 0. \tag{65}$$

Using the fact that E is an increasing function of c, then it leads to

$$E_\mathrm{B} - E_0 \geqslant 0. \tag{66}$$

Thus, by varying the concentration of the phases, the effective modulus can be improved over the homogeneous situation by concentrating the phase with the greater Young's modulus farther from the y axis.

5. Conclusions

The tension, bending and flexure of cylinders with functionally graded cross-sections are examined. It has been demonstrated that, as the elastic moduli are convex functions of the volume fractions, the effective Young's modulus in simple tension achieves its minimum for the homogeneous distribution of the phases.

Further, by concentrating the phases with larger Young's moduli farther from the axis of bending the effective Young's modulus will be greater than the homogeneous modulus. In other words, if a structural member is required to be just in tension, any grading of the phases will result in an improvement in the performance. If the component is required to bend or flex, then concentrating the phases with larger Young's moduli away from the axis of bending will lead to less deformation for a given bending moment.

References

Aboudi, J., Pindera, M.-J., Arnold, S.M., 1995. A coupled higher-order theory for functionally graded composites with partial homogenization. Comp. Engng. 5, 771–792.

Benveniste, Y., 1987. A new approach to the application of Mori-Tanaka's theory in composite materials. Mech. Mat. 6, 147–157.

Casey, J., Kaplan, A., 1997. Adhemar-Jean-Claude Barre de Saint-Venant. 23 August 1797–6 January 1886. Math. Mech. Solids 2, 371-378.

Christensen, R.M., 1990. A critical evaluation for a class of micro-mechanics models. J. Mech. Phys. Solids 38, 379–404.

Delale, F., Erdogan, F., 1988. The interface crack in a non homogeneous medium. Int. J. Engng. Sci. 28, 559–568.

Ely, J.F., Zienkiewicz, O.C., 1960. Torsion of compound bars-a relaxation solution. Int. J. Mech. Sci. 1, 356–365.

Erdogan, F., 1995. Fracture mechanics of functionally graded materials. Comp. Engng 5, 753–770.

Erdogan, F., Ozturk, M., 1992. Diffusion problems in bonded non homogeneous materials with an interface cut. Int. J. Engng. Sci. 30, 1507–1523.

Ferrari, M., 1992. Anisotropic layers with through-thickness thermal and material variations. J. Therm. Stresses 15, 439–445.

Ferrari, M., 1994. Composite homogenization via the poly-inclusion method. Comp. Engng. 4, 37–45.

Gurtin, M.E., Truesdell, C.A., 1984. The linear theory of elasticity. In: Truesdell, C.A. (Ed.), Mechanics of Solids II, Springer, Berlin.

Hill, R., 1963. Elastic properties of reinforced solids: some theoretical principles. J. Mech. Phys. Solids 11, 357–372.

Koizumi, M., 1992. The concept of FGM, Proceedings of the Second International Symposium on Functionally Gradient Materials at the Third International Ceramic Science and Technology Congress, San Francisco.

Lutz, M.P., Ferrari, M., 1993. Compression of a sphere with radially varying elastic moduli. Comp. Engng. 3, 873–884.

Mushkhelishvili, N.I., 1963. Some Basic problems of the Mathematical Theory of Elasticity. Nordhoff, Groningen.

Nadeau, J.C., Ferrari, M., 1998. Microstructural optimization of a functionally Graded transversely isotropic layer. Mech. Mat., submitted for publication.

Niino, M., Hirai, T., Watanabe, R., 1987. Functionally gradient materials as thermal barrier for space plane. J. Jpn. Soc. Comp. Mater. 13, 257–264.

Reissner, E., 1964. Note on the problem of St. Venant flexure. ZAMP 15, 198–200.

Reiter, T., Dvorak, G.J., Tvergaard, V., 1997. Micromechanical models for graded composite materials. J. Mech. Phys. Solids 45, 1281–1302.

Rockafellar, R.T., 1970. Convex Analysis. Princeton University Press, NJ.

Rooney, F.J., Ferrari, M., 1995. Torsion and flexure of inhomogeneous elements. Comp. Engng. 5, 901–911.

Rooney, F.J., Ferrari, M., 1999. On the St. Venant problems for inhomogeneous circular bars. J. Appl. Mech., submitted for publication.

Schile, R.D., 1962. Bending of non homogeneous bars. Int. J. Mech. Sci. 5, 439–445.

Schile, R.D., Sierakowski, R.L., 1965. On the St. Venant problem for a non homogeneous elastic material. Quart. Appl. Math. 23, 19–25.

Zuiker, J.R., Dvorak, G.J., 1994. The effective properties of functionally graded composites-I. Extension of the Mori-Tanaka method to linearly varying fields. Comp. Engng. 4, 19–35.

Zuiker, J.R., Dvorak, G.J., 1994. The effective properties of composite materials with constant reinforcement density by the linear Mori-Tanaka method. J. Engng. Mat. Tech. 116, 428–437.